第 9 章

多元函数微分法及其应用

一、基 本 内 容

1. 多元函数的基本概念
   (1) 平面点集, *n 维空间;
   (2) 多元函数的概念;
   (3) 多元函数的极限;
   (4) 多元函数的连续性.
2. 偏导数
   (1) 偏导数的定义及其计算法;
   (2) 高阶偏导数.
3. 全微分
   (1) 全微分的定义;
   *(2) 全微分在近似计算中的应用.
4. 多元复合函数的求导法则
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   (2) 多元函数与多元函数复合的情形;
   (3) 其他情形, 全微分形式不变性.
5. 隐函数的求导公式
   (1) 一个方程的情形: 隐函数存在定理 1、隐函数存在定理 2;
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6. 多元函数微分学的几何应用
   (1) 一元向量值函数及其导数;
   (2) 空间曲线的切线与法平面;
   (3) 曲面的切平面与法线.
7. 方向导数与梯度
   (1) 方向导数;
   (2) 梯度.
8. 多元函数的极值及其求法
   (1) 多元函数的极值及最大值与最小值;
   (2) 条件极值, 拉格朗日乘数法.

*9. 二元函数的泰勒公式

(1) 二元函数的泰勒公式;

(2) 极值充分条件的证明.

*10. 最小二乘法

二、基 本 要 求

1. 理解多元函数的概念和二元函数的几何意义.
2. 了解二元函数的极限与连续性的概念.
3. 理解多元函数偏导数和全微分的概念, 会求全微分, 了解全微分存在的必要条件, 了解全微分形式的不变性.
4. 掌握多元复合函数偏导数的求法.
5. 会求隐函数（包括由方程组确定的隐函数）的偏导数.
6. 了解曲线的切线和法平面及曲面的切平面和法线的概念, 会求它们的方程.
7. 理解方向导数与梯度的概念并掌握其计算方法.
8. 了解二元函数的二阶泰勒公式.
9. 理解多元函数的极值和条件极值的概念, 掌握多元函数极值存在的必要条件, 了解二元函数极值存在的充分条件, 会求二元函数的极值, 会用拉格朗日乘数法求条件极值.
10. 了解多元函数最大值和最小值的概念, 并会解决一些简单的应用问题.

三、习 题 解 答

习题 9-1

1. 判定下列平面点集中哪些是开集、闭集、区域、有界集、无界集, 并分别指出它们的聚点所成的点集（称为导集）和边界.

(1) $\{(x,y)|x\neq 0, y\neq 0\}$; (2) $\{(x,y)|1<x^2+y^2\leqslant 4\}$;

(3) $\{(x,y)|y>x^2\}$; (4) $\{(x,y)|x^2+(y-1)^2\geqslant 1\}\cap\{(x,y)|x^2+(y-2)^2\leqslant 4\}$.

解 (1) 此集合是开集, 无界集; 导集为 R^2, 边界为 $\left\{(x,y)\middle|x=0\text{或}y=0\right\}$.

(2) 此集合既非开集也非闭集, 是有界集; 导集为 $\{(x,y)|1\leqslant x^2+y^2\leqslant 4\}$, 边界为 $\{(x,y)|x^2+y^2=1\}\cup\{(x,y)|x^2+y^2=4\}$.

(3) 此集合是开集, 区域, 无界集; 导集为 $\{(x,y)|y\geqslant x^2\}$, 边界为 $\{(x,y)|y=x^2\}$.

(4) 此集合为闭集, 有界集; 导集为集合本身, 边界为

$$\{(x,y)|x^2+(y-1)^2=1\}\cup\{(x,y)|x^2+(y-2)^2=4\}.$$

2. 已知函数 $f(x,y)=x^2+y^2-xy\tan\dfrac{x}{y}$, 试求 $f(tx,ty)$.

解 $f(tx,ty)=(tx)^2+(ty)^2-(tx)(ty)\tan\dfrac{tx}{ty}=t^2\left(x^2+y^2-xy\tan\dfrac{x}{y}\right)=t^2 f(x,y)$.

3. 试证函数 $F(x,y)=\ln x\cdot\ln y$ 满足关系式

$$F(xy,uv)=F(x,u)+F(x,v)+F(y,u)+F(y,v).$$

证明 $F(xy,uv)=\ln(xy)\cdot\ln(uv)=(\ln x+\ln y)\cdot(\ln u+\ln v)$

$$=\ln x\cdot\ln u+\ln x\cdot\ln v+\ln y\cdot\ln u+\ln y\cdot\ln v$$

$$=F(x,u)+F(x,v)+F(y,u)+F(y,v).$$

4. 已知函数 $f(u,v,w)=u^w+w^{u+v}$，试求 $f(x+y,x-y,xy)$.

解 $f(x+y,x-y,xy)=(x+y)^{xy}+(xy)^{x+y+x-y}=(x+y)^{xy}+(xy)^{2x}$.

5. 求下列各函数的定义域.

(1) $z=\ln(y^2-2x+1)$; (2) $z=\dfrac{1}{\sqrt{x+y}}+\dfrac{1}{\sqrt{x-y}}$;

(3) $z=\sqrt{x-\sqrt{y}}$; (4) $z=\ln(y-x)+\dfrac{\sqrt{x}}{\sqrt{1-x^2-y^2}}$;

(5) $u=\sqrt{R^2-x^2-y^2-z^2}+\dfrac{1}{\sqrt{x^2+y^2+z^2-r^2}}(R>r>0)$;

(6) $u=\arccos\dfrac{z}{\sqrt{x^2+y^2}}$.

注：求多元函数的定义域与求一元函数的定义域相类似，就是先求出构成该函数的各个简单函数的定义域，然后再求出它们的交集，即为所求的定义域.

解 (1) $\{(x,y)|y^2-2x+1>0\}$.

(2) $\{(x,y)|x+y>0,x-y>0\}$.

(3) $\{(x,y)|x\geqslant 0,y\geqslant 0,x^2\geqslant y\}$.

(4) $\{(x,y)|y-x>0,x\geqslant 0,x^2+y^2<1\}$.

(5) $\{(x,y,z)|r^2<x^2+y^2+z^2\leqslant R^2\}$.

(6) $\{(x,y,z)|x^2+y^2\geqslant z^2,x^2+y^2\neq 0\}$.

6. 求下列各极限.

(1) $\lim\limits_{(x,y)\to(0,1)}\dfrac{1-xy}{x^2+y^2}$; (2) $\lim\limits_{(x,y)\to(1,0)}\dfrac{\ln(x+\mathrm{e}^y)}{\sqrt{x^2+y^2}}$; (3) $\lim\limits_{(x,y)\to(0,0)}\dfrac{2-\sqrt{xy+4}}{xy}$;

(4) $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy}{\sqrt{2-\mathrm{e}^{xy}}-1}$; (5) $\lim\limits_{(x,y)\to(2,0)}\dfrac{\tan(xy)}{y}$; (6) $\lim\limits_{(x,y)\to(0,0)}\dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)\mathrm{e}^{x^2y^2}}$.

解 (1) $\lim\limits_{(x,y)\to(0,1)}\dfrac{1-xy}{x^2+y^2}=\dfrac{1-0}{0+1}=1.$

(2) $\lim\limits_{(x,y)\to(1,0)}\dfrac{\ln(x+\mathrm{e}^y)}{\sqrt{x^2+y^2}}=\dfrac{\ln(1+\mathrm{e}^0)}{1}=\ln 2.$

注：上述两小题是利用多元初等函数在其定义域是连续的这个性质来求极限的，即函数在该点

的极限值等于函数在该点的函数值.

(3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2-\sqrt{xy+4}}{xy} = \lim\limits_{(x,y)\to(0,0)} \dfrac{4-(xy+4)}{xy(2+\sqrt{xy+4})} = \lim\limits_{(x,y)\to(0,0)} \dfrac{-1}{2+\sqrt{xy+4}} = -\dfrac{1}{4}.$

(4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{2-e^{xy}}-1} = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{1-e^{xy}}(\sqrt{2-e^{xy}}+1) = -1\cdot 2 = -2.$

注: 此题先运用了分母有理化, 然后又利用了当$(x,y)\to(0,0)$时, $e^{xy}-1 \sim xy$.

(5) $\lim\limits_{(x,y)\to(2,0)} \dfrac{\tan(xy)}{y} = \lim\limits_{(x,y)\to(2,0)} \dfrac{\tan(xy)}{xy}\cdot x = 1\cdot 2 = 2.$

注: 本题利用了当$(x,y)\to(2,0)$时, $\tan(xy)\sim xy$.

(6) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)e^{x^2y^2}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{\frac{1}{2}(x^2+y^2)^2}{(x^2+y^2)e^{x^2y^2}} = \dfrac{1}{2}\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{e^{x^2y^2}} = \dfrac{1}{2}\cdot 0 = 0.$

注: 本题利用了当$(x,y)\to(0,0)$时, $1-\cos(x^2+y^2)\sim \dfrac{1}{2}(x^2+y^2)^2$.

***7**. 证明下列极限不存在.

(1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x-y}$; (2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2y^2+(x-y)^2}$.

注: 证明极限 $\lim\limits_{P\to P_0} f(P)$ 不存在常用的方法有: (1) 找两条不同的路径, 使得点P沿着这两条路径趋于P_0时, $f(P)$的极限存在但不相等; (2) 找一条特殊的路径, 使得点P沿着这条路径趋于P_0时, $f(P)$的极限不存在. 本题采用第一种方法.

解 (1) 当点(x,y)沿直线$y=\dfrac{1}{2}x, y=2x$这两条路径趋于$(0,0)$时, 有

$$\lim_{\substack{(x,y)\to(0,0)\\ y=\frac{1}{2}x}} \frac{x+y}{x-y} = \lim_{x\to 0}\frac{x+\frac{1}{2}x}{x-\frac{1}{2}x} = 3, \qquad \lim_{\substack{(x,y)\to(0,0)\\ y=2x}} \frac{x+y}{x-y} = \lim_{x\to 0}\frac{x+2x}{x-2x} = -3.$$

故所求极限不存在.

(2) 当点(x,y)沿直线$y=x, y=-x$这两条路径趋于$(0,0)$时, 有

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x^2y^2}{x^2y^2+(x-y)^2} = \lim_{x\to 0}\frac{x^4}{x^4+0} = 1,$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=-x}} \frac{x^2y^2}{x^2y^2+(x-y)^2} = \lim_{x\to 0}\frac{x^4}{x^4+4x^2} = \lim_{x\to 0}\frac{x^2}{x^2+4} = 0.$$

故所求极限不存在.

8. 函数 $z=\dfrac{y^2+2x}{y^2-2x}$ 在何处是间断的?

解 这个函数的定义域是$\{(x,y)\,|\,y^2-2x\neq 0\}$, 也就是函数在曲线$y^2=2x$上没有定义, 因此曲线$y^2=2x$上的各点均为函数的间断点.

***9**. 证明 $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy}{\sqrt{x^2+y^2}}=0$.

证明 $\forall\varepsilon>0$.

因为 $\left|\dfrac{xy}{\sqrt{x^2+y^2}}-0\right|\leqslant\dfrac{\frac{1}{2}(x^2+y^2)}{\sqrt{x^2+y^2}}=\dfrac{1}{2}\sqrt{x^2+y^2}$，要使 $\left|\dfrac{xy}{\sqrt{x^2+y^2}}-0\right|<\varepsilon$，只要 $\sqrt{x^2+y^2}<2\varepsilon$，

所以取 $\delta=2\varepsilon$，则当 $0<\sqrt{x^2+y^2}<\delta$ 时，有 $\left|\dfrac{xy}{\sqrt{x^2+y^2}}-0\right|<\varepsilon$ 成立，即 $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy}{\sqrt{x^2+y^2}}=0.$

***10**. 设 $F(x,y)=f(x)$，$f(x)$ 在 x_0 处连续. 证明：对任意 $y_0\in\mathrm{R}, F(x,y)$ 在 (x_0,y_0) 处连续.

证明 取点 $P_0(x_0,y_0)\in\mathrm{R}^2$，因为 $f(x)$ 在 x_0 处连续，所以 $\forall\varepsilon>0$，$\exists\delta>0$, 当 $|x-x_0|<\delta$ 时，有 $|f(x)-f(x_0)|<\varepsilon$. 所以当 $P(x,y)\in U(P_0,\delta)$ 时，有 $|x-x_0|<\rho(P,P_0)<\delta$，从而有 $|F(x,y)-F(x_0,y_0)|=|f(x)-f(x_0)|<\varepsilon$，即 $F(x,y)$ 在 (x_0,y_0) 处连续.

习题 9-2

1. 求下列函数的偏导数.

(1) $z=x^3y-y^3x$;　(2) $s=\dfrac{u^2+v^2}{uv}$;　(3) $z=\sqrt{\ln(xy)}$;

(4) $z=\sin(xy)+\cos^2(xy)$;　(5) $z=\ln\tan\dfrac{x}{y}$;　(6) $z=(1+xy)^y$;

(7) $u=x^{\frac{y}{z}}$;　(8) $u=\arctan(x-y)^z$.

解 (1) $\dfrac{\partial z}{\partial x}=3x^2y-y^3, \dfrac{\partial z}{\partial y}=x^3-3y^2x$.

(2) 利用商的求导法则，得

$$\frac{\partial s}{\partial u}=\frac{2u\cdot uv-(u^2+v^2)v}{(uv)^2}=\frac{1}{v}-\frac{v}{u^2},\quad \frac{\partial s}{\partial v}=\frac{2v\cdot uv-(u^2+v^2)u}{(uv)^2}=\frac{1}{u}-\frac{u}{v^2}.$$

(3) $\dfrac{\partial z}{\partial x}=\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{\ln(xy)}}\cdot\dfrac{1}{xy}\cdot y=\dfrac{1}{2x\sqrt{\ln(xy)}}$，$\dfrac{\partial z}{\partial y}=\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{\ln(xy)}}\cdot\dfrac{1}{xy}\cdot x=\dfrac{1}{2y\sqrt{\ln(xy)}}$.

(4) $\dfrac{\partial z}{\partial x}=y\cos(xy)+2\cos(xy)\cdot[-\sin(xy)]\cdot y=y[\cos(xy)-\sin(2xy)]$,

$\dfrac{\partial z}{\partial y}=x\cos(xy)+2\cos(xy)\cdot[-\sin(xy)]\cdot x=x[\cos(xy)-\sin(2xy)]$.

(5) $\dfrac{\partial z}{\partial x}=\dfrac{1}{\tan\frac{x}{y}}\cdot\sec^2\dfrac{x}{y}\cdot\dfrac{1}{y}=\dfrac{1}{y\sin\frac{x}{y}\cos\frac{x}{y}}=\dfrac{2}{y}\csc\dfrac{2x}{y}$,

$\dfrac{\partial z}{\partial y}=\dfrac{1}{\tan\frac{x}{y}}\cdot\sec^2\dfrac{x}{y}\cdot\left(-\dfrac{x}{y^2}\right)=-\dfrac{x}{y^2\sin\frac{x}{y}\cos\frac{x}{y}}=-\dfrac{2x}{y^2}\csc\dfrac{2x}{y}$.

(6) $\frac{\partial z}{\partial x}=y^2(1+xy)^{y-1}$，$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}(e^{y\ln(1+xy)})=(1+xy)^y\left[\ln(1+xy)+\frac{xy}{1+xy}\right]$.

(7) $\frac{\partial u}{\partial x}=\frac{y}{z}x^{\frac{y}{z}-1}, \frac{\partial u}{\partial y}=\frac{1}{z}x^{\frac{y}{z}}\ln x, \frac{\partial u}{\partial z}=-\frac{y}{z^2}x^{\frac{y}{z}}\ln x$.

(8) $\frac{\partial u}{\partial x}=\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}, \frac{\partial u}{\partial y}=-\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$，$\frac{\partial u}{\partial z}=\frac{(x-y)^z\ln(x-y)}{1+(x-y)^{2z}}$.

2. 设$T=2\pi\sqrt{l/g}$，求证$l\frac{\partial T}{\partial l}+g\frac{\partial T}{\partial g}=0$.

证明 因为$\frac{\partial T}{\partial l}=2\pi.\frac{1}{2}.\frac{1}{\sqrt{l/g}}.\frac{1}{g}=\frac{\pi}{\sqrt{gl}}$，$\frac{\partial T}{\partial g}=2\pi\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{l/g}}\cdot\left(-\frac{l}{g^2}\right)=-\frac{\pi\sqrt{l}}{g\sqrt{g}}$，所以

$l\frac{\partial T}{\partial l}+g\frac{\partial T}{\partial g}=\pi\sqrt{l/g}-\pi\sqrt{l/g}=0$

3. 设$z=e^{-\left(\frac{1}{x}+\frac{1}{y}\right)}$，求证$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=2z$.

证明 因为$\frac{\partial z}{\partial x}=\frac{1}{x^2}e^{-\left(\frac{1}{x}+\frac{1}{y}\right)}, \frac{\partial z}{\partial y}=\frac{1}{y^2}e^{-\left(\frac{1}{x}+\frac{1}{y}\right)}$，所以有

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=e^{-\left(\frac{1}{x}+\frac{1}{y}\right)}+e^{-\left(\frac{1}{x}+\frac{1}{y}\right)}=2z.$$

4. 设$f(x,y)=x+(y-1)\arcsin\sqrt{\frac{x}{y}}$，求$f_x(x,1)$.

解 $f_x(x,y)=1+\frac{y-1}{\sqrt{1-\frac{x}{y}}}.\frac{1}{2\sqrt{\frac{x}{y}}}.\frac{1}{y}$，所以$f_x(x,1)=1$.

5. 曲线$\begin{cases}z=\frac{x^2+y^2}{4}\\ y=4\end{cases}$在点$(2,4,5)$处的切线对于$x$轴的倾角是多少?

解 设$z=f(x,y)$，按偏导数的几何意义，$f_x(2,4)$就是曲线在点$(2,4,5)$处的切线对于x轴的斜率，而$f_x(2,4)=\frac{1}{2}x\Big|_{x=2}=1$，所以所求倾角为$\frac{\pi}{4}$.

6. 求下列函数的$\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial y^2}$和$\frac{\partial^2 z}{\partial x\partial y}$.

(1) $z=x^4+y^4-4x^2y^2$；　(2) $z=\arctan\frac{y}{x}$；　(3) $z=y^x$.

解 (1)$\frac{\partial z}{\partial x}=4x^3-8xy^2$，$\frac{\partial z}{\partial y}=4y^3-8x^2y$，

$$\frac{\partial^2 z}{\partial x^2}=12x^2-8y^2,\quad \frac{\partial^2 z}{\partial y^2}=12y^2-8x^2,\quad \frac{\partial^2 z}{\partial x\partial y}=-16xy.$$

(2) $\dfrac{\partial z}{\partial x}=\dfrac{1}{1+\left(\dfrac{y}{x}\right)^2}\cdot\left(-\dfrac{y}{x^2}\right)=-\dfrac{y}{x^2+y^2}$, $\dfrac{\partial z}{\partial y}=\dfrac{1}{1+\left(\dfrac{y}{x}\right)^2}\cdot\dfrac{1}{x}=\dfrac{x}{x^2+y^2}$,

$$\frac{\partial^2 z}{\partial x^2}=\frac{2xy}{(x^2+y^2)^2},\quad \frac{\partial^2 z}{\partial y^2}=-\frac{2xy}{(x^2+y^2)^2},\quad \frac{\partial^2 z}{\partial x\partial y}=-\frac{x^2+y^2-y.2y}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}.$$

(3) $\dfrac{\partial z}{\partial x}=y^x\ln y$, $\dfrac{\partial z}{\partial y}=xy^{x-1}$,

$$\frac{\partial^2 z}{\partial x^2}=y^x\ln^2 y,\quad \frac{\partial^2 z}{\partial y^2}=x(x-1)y^{x-2},\quad \frac{\partial^2 z}{\partial x\partial y}=y^{x-1}(1+x\ln y).$$

7. 设 $f(x,y,z)=xy^2+yz^2+zx^2$，求 $f_{xx}(0,0,1), f_{xz}(1,0,2), f_{yz}(0,-1,0)$ 和 $f_{zzx}(2,0,1)$.

解 因为 $f_x=y^2+2xz$，$f_y=2xy+z^2$，$f_z=2zy+x^2$，$f_{xx}=2z$，$f_{xz}=2x$，$f_{yz}=2z$，$f_{zz}=2y$，$f_{zzx}=0$，所以有 $f_{xx}(0,0,1)=2$，$f_{xz}(1,0,2)=2$，$f_{yz}(0,-1,0)=0$，$f_{zzx}(2,0,1)=0$.

8. 设 $z=x\ln(xy)$，求 $\dfrac{\partial^3 z}{\partial x^2\partial y}$ 和 $\dfrac{\partial^3 z}{\partial x\partial y^2}$.

解 $\dfrac{\partial z}{\partial x}=\ln(xy)+x.\dfrac{y}{xy}=1+\ln(xy)$，$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{y}{xy}=\dfrac{1}{x}$，$\dfrac{\partial^3 z}{\partial x^2\partial y}=0$，

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{x}{xy}=\frac{1}{y},\quad \frac{\partial^3 z}{\partial x\partial y^2}=-\frac{1}{y^2}.$$

9. 验证:

(1) $y=\mathrm{e}^{-kn^2t}\sin nx$ 满足 $\dfrac{\partial y}{\partial t}=k\dfrac{\partial^2 y}{\partial x^2}$.

(2) $r=\sqrt{x^2+y^2+z^2}$ 满足 $\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}+\dfrac{\partial^2 r}{\partial z^2}=\dfrac{2}{r}$.

证明 (1) 因为 $\dfrac{\partial y}{\partial t}=-kn^2\mathrm{e}^{-kn^2t}\sin nx$，$\dfrac{\partial y}{\partial x}=n\mathrm{e}^{-kn^2t}\cos nx$，$\dfrac{\partial^2 y}{\partial x^2}=-n^2\mathrm{e}^{-kn^2t}\sin nx$，

所以有 $\dfrac{\partial y}{\partial t}=k\dfrac{\partial^2 y}{\partial x^2}$.

(2) $\dfrac{\partial r}{\partial x}=\dfrac{x}{\sqrt{x^2+y^2+z^2}}=\dfrac{x}{r}$，$\dfrac{\partial^2 r}{\partial x^2}=\dfrac{r-x\dfrac{\partial r}{\partial x}}{r^2}=\dfrac{r^2-x^2}{r^3}$，由于函数关于自变量对称，得

$$\frac{\partial^2 r}{\partial y^2}=\frac{r^2-y^2}{r^3},\quad \frac{\partial^2 r}{\partial z^2}=\frac{r^2-z^2}{r^3},$$

所以有 $\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}+\dfrac{\partial^2 r}{\partial z^2}=\dfrac{3r^2-x^2-y^2-z^2}{r^3}=\dfrac{2}{r}$.

习题 9-3

1. 求下列函数的全微分.

(1) $z = xy + \frac{x}{y}$;　(2) $z = e^{\frac{y}{x}}$;　(3) $z = \frac{y}{\sqrt{x^2 + y^2}}$;　(4) $u = x^{yz}$.

解 (1) 因为 $\frac{\partial z}{\partial x} = y + \frac{1}{y}$, $\frac{\partial z}{\partial y} = x - \frac{x}{y^2}$, 所以

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \left(y + \frac{1}{y}\right)dx + \left(x - \frac{x}{y^2}\right)dy .$$

(2) 因为 $\frac{\partial z}{\partial x} = -\frac{y}{x^2}e^{\frac{y}{x}}$, $\frac{\partial z}{\partial y} = \frac{1}{x}e^{\frac{y}{x}}$, 所以

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = -\frac{1}{x^2}e^{\frac{y}{x}}(ydx - xdy) .$$

(3) 因为 $\frac{\partial z}{\partial x} = -\frac{y}{x^2 + y^2} \cdot \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2 + y^2}} = -\frac{xy}{(x^2 + y^2)^{\frac{3}{2}}}$,

$$\frac{\partial z}{\partial y} = \frac{\sqrt{x^2 + y^2} - y.\frac{y}{\sqrt{x^2 + y^2}}}{x^2 + y^2} = \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}}, \text{ 所以}$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = -\frac{x}{(x^2 + y^2)^{\frac{3}{2}}}(ydx - xdy) .$$

(4) 因为 $\frac{\partial u}{\partial x} = yzx^{yz-1}$, $\frac{\partial u}{\partial y} = zx^{yz}\ln x$, $\frac{\partial u}{\partial z} = yx^{yz}\ln x$, 所以

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = yzx^{yz-1}dx + zx^{yz}\ln xdy + yx^{yz}\ln xdz .$$

2. 求函数 $z = \ln(1 + x^2 + y^2)$ 当 $x = 1, y = 2$ 时的全微分.

解 因为 $\frac{\partial z}{\partial x} = \frac{2x}{1 + x^2 + y^2}$, $\frac{\partial z}{\partial y} = \frac{2y}{1 + x^2 + y^2}$, 所以 $\left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\y=2}} = \frac{1}{3}, \left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=2}} = \frac{2}{3}$, 从而所求全微分

$$\left.dz\right|_{\substack{x=1\\y=2}} = \frac{1}{3}dx + \frac{2}{3}dy .$$

3. 求函数 $z = \frac{y}{x}$ 当 $x = 2, y = 1, \Delta x = 0.1, \Delta y = -0.2$ 时的全增量和全微分.

解 $\Delta z = \frac{y + \Delta y}{x + \Delta x} - \frac{y}{x}$, $dz = -\frac{y}{x^2}\Delta x + \frac{1}{x}\Delta y$, 所以当 $x = 2, y = 1, \Delta x = 0.1, \Delta y = -0.2$ 时的全增量 $\Delta z = \frac{1 - 0.2}{2 + 0.1} - \frac{1}{2} = -0.119$, 全微分 $dz = -\frac{0.1}{4} + \frac{1}{2} \times (-0.2) = -0.125$.

4. 求函数 $z=\mathrm{e}^{xy}$ 当 $x=1, y=1, \Delta x=0.15, \Delta y=0.1$ 时的全微分.

解 因为 $\mathrm{d}z=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y=y\mathrm{e}^{xy}\Delta x+x\mathrm{e}^{xy}\Delta y$,

所以当 $x=1, y=1, \Delta x=0.15, \Delta y=0.1$ 时的全微分 $\mathrm{d}z=\mathrm{e}\cdot 0.15+\mathrm{e}\cdot 0.1=0.25\mathrm{e}$.

5. 考虑二元函数 $f(x,y)$ 的下面四条性质:

(1) $f(x,y)$ 在点 (x_0,y_0) 连续;

(2) $f_x(x,y), f_y(x,y)$ 在点 (x_0,y_0) 连续;

(3) $f(x,y)$ 在点 (x_0,y_0) 可微分;

(4) $f_x(x_0,y_0), f_y(x_0,y_0)$ 存在.

若用“ $P\Rightarrow Q$ ”表示可由性质 P 推出性质 Q, 则下列四个选项中正确的是（　　）.

A．$(2)\Rightarrow(3)\Rightarrow(1)$　　B．$(3)\Rightarrow(2)\Rightarrow(1)$

C．$(3)\Rightarrow(4)\Rightarrow(1)$　　D．$(3)\Rightarrow(1)\Rightarrow(4)$

解 由已证定理知, 二元函数可微分可推得二元函数偏导数存在且连续, 而偏导数连续可推得函数可微分, 也即得到 $(2)\Rightarrow(3)\Rightarrow(1)$、$(4)$, 所以选 A.

***6**. 计算 $\sqrt{(1.02)^3+(1.97)^3}$ 的近似值.

解 设 $z=\sqrt{x^3+y^3}$, 则 $z_x=\frac{3x^2}{2\sqrt{x^3+y^3}}$, $z_y=\frac{3y^2}{2\sqrt{x^3+y^3}}$.

所以 $\sqrt{(x+\Delta x)^3+(y+\Delta y)^3}=\sqrt{x^3+y^3}+\Delta z$

$$\approx\sqrt{x^3+y^3}+\mathrm{d}z=\sqrt{x^3+y^3}+\frac{3x^2\Delta x+3y^2\Delta y}{2\sqrt{x^3+y^3}}.$$

取 $x=1, y=2, \Delta x=0.02, \Delta y=-0.03$, 可得

$$\sqrt{(1.02)^3+(1.97)^3}\approx\sqrt{1+2^3}+\frac{3\times 1\times 0.02+3\times 2^2\times(-0.03)}{2\sqrt{1+2^3}}=2.95.$$

***7**. 计算 $(1.97)^{1.05}$ 的近似值（ $\ln 2=0.693$ ）.

解 设 $z=x^y$, 则 $z_x=yx^{y-1}, z_y=x^y\ln x$. 所以

$$(x+\Delta x)^{y+\Delta y}=x^y+\Delta z\approx x^y+\mathrm{d}z=x^y+yx^{y-1}\cdot\Delta x+x^y\ln x\cdot\Delta y.$$

取 $x=2, y=1, \Delta x=-0.03, \Delta y=0.05$, 可得

$$(1.97)^{1.05}\approx 2-0.03+2\ln 2\cdot 0.05\approx 2.039.$$

***8**. 已知边长为 $x=6\mathrm{m}$ 与 $y=8\mathrm{m}$ 的矩形, 如果 x 边增加 $5\mathrm{cm}$ 而 y 边减少 $10\mathrm{cm}$, 问这个矩形的对角线的近似变化怎样?

解 矩形的对角线的长为 $z=\sqrt{x^2+y^2}$, 则

$$\Delta z\approx\mathrm{d}z=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y=\frac{1}{\sqrt{x^2+y^2}}(x\Delta x+y\Delta y).$$

所以当 $x=6, y=8, \Delta x=0.05, \Delta y=-0.1$ 时，对角线的变化

$$\Delta z \approx \frac{1}{\sqrt{6^2+8^2}}(6\times 0.05-8\times 0.1)=-0.05,$$

也即这个矩形的对角线的长减少大约 5cm .

***9**. 设有一无盖圆柱形容器，容器的壁与底的厚度均为 0.1cm，内高为 20cm，内半径为 4cm．求容器外壳体积的近似值.

解 圆柱形的体积 $V=\pi r^2 h$，所求容器外壳体积就是圆柱体体积的增量 ΔV，而

$$\Delta V \approx \mathrm{d}V=\frac{\partial V}{\partial r}\Delta r+\frac{\partial V}{\partial h}\Delta h=2\pi rh\Delta r+\pi r^2\Delta h.$$

则当 $r=4, h=20, \Delta r=\Delta h=0.1$ 时，$\Delta V \approx 2\times 3.14\times 4\times 20\times 0.1+3.1\times 4^2\times 0.1\approx 55.3$，即容器外壳的体积大约是 $55.3\mathrm{cm}^3$.

***10**. 设有直角三角形，测得其两直角边的长分别为 (7 ± 0.1)cm 和 (24 ± 0.1)cm．试求利用上述二值来计算斜边长度时的绝对误差.

解 设两直角边长度分别为 x 和 y，则斜边长度为 $z=\sqrt{x^2+y^2}$，

$$\begin{aligned}|\Delta z| \approx |\mathrm{d}z| &= \left|\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y\right| \leqslant \left|\frac{\partial z}{\partial x}\right||\Delta x|+\left|\frac{\partial z}{\partial y}\right||\Delta y| \\ &= \frac{1}{\sqrt{x^2+y^2}}(x|\Delta x|+y|\Delta y|) \leqslant \frac{1}{\sqrt{x^2+y^2}}(x\delta_x+y\delta_y)\end{aligned},$$

可得 $\delta_z=\frac{1}{\sqrt{x^2+y^2}}(x\delta_x+y\delta_y)$．则当 $x=7, y=24, \delta_x=0.1, \delta_y=0.1$ 时，

$$\delta_z=\frac{1}{\sqrt{7^2+24^2}}(7\times 0.1+24\times 0.1)\approx 0.124.$$

即计算斜边长度的绝对误差约为 0.124cm .

***11**. 测得一块三角形土地的两边边长分别为 (63 ± 0.1)m 和 (78 ± 0.1)m，这两边的夹角为 $60^\circ\pm 1^\circ$．试求三角形面积的近似值，并求其绝对误差和相对误差.

解 设三角形的两边长分别为 a 和 b，它们的夹角为 θ，则三角形的面积为 $S=\frac{1}{2}ab\sin\theta$，

$$\begin{aligned}|\Delta S| \approx |\mathrm{d}S| &= \left|\frac{\partial S}{\partial a}\Delta a+\frac{\partial S}{\partial b}\Delta b+\frac{\partial S}{\partial \theta}\Delta\theta\right| \leqslant \left|\frac{\partial S}{\partial a}\right||\Delta a|+\left|\frac{\partial S}{\partial b}\right||\Delta b|+\left|\frac{\partial S}{\partial \theta}\right||\Delta\theta| \\ &= \frac{1}{2}b\sin\theta|\Delta a|+\frac{1}{2}a\sin\theta|\Delta b|+\frac{1}{2}ab\cos\theta|\Delta\theta| \\ &\leqslant \frac{1}{2}b\sin\theta\delta_a+\frac{1}{2}a\sin\theta\delta_b+\frac{1}{2}ab\cos\theta\delta_\theta,\end{aligned}$$

可得

$$\delta_S=\frac{1}{2}b\sin\theta\delta_a+\frac{1}{2}a\sin\theta\delta_b+\frac{1}{2}ab\cos\theta\delta_\theta,$$

则当 $a=63, b=78, \theta=\frac{\pi}{3}, \delta_a=0.1, \delta_b=0.1, \delta_\theta=\frac{\pi}{180}$ 时，三角形面积的近似值为

$$S=\frac{1}{2}\times 63\times 78\cdot\sin\frac{\pi}{3}\approx 2127.8(\mathrm{m}^2),$$

绝对误差为

$$\delta_S=\frac{1}{2}\times 78\times\frac{\sqrt{3}}{2}\times 0.1+\frac{1}{2}\times 63\times\frac{\sqrt{3}}{2}\times 0.1+\frac{1}{2}\times 63\times 78\times\frac{\sqrt{3}}{2}\times\frac{1}{2}\times\frac{\pi}{180}\approx 27.6(\mathrm{m}^2),$$

相对误差为

$$\frac{\delta_S}{S}\approx\frac{27.6}{2127.8}\approx 1.30\%.$$

***12.** 利用全微分证明：两数之和的绝对误差等于它们各自的绝对误差之和.

证明 设 $u=x+y$，则

$$|\Delta u|\approx|\mathrm{d}u|=\left|\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y\right|=|\Delta x+\Delta y|\leqslant|\Delta x|+|\Delta y|\leqslant\delta_x+\delta_y,$$

于是有 $\delta_u=\delta_x+\delta_y$，即两数之和的绝对误差等于它们各自的绝对误差之和.

***13.** 利用全微分证明：乘积的相对误差等于各因子的相对误差之和；商的相对误差等于被除数及除数的相对误差之和.

证明 设 $u=xy, v=\frac{x}{y}$，则

$$|\Delta u|\approx|\mathrm{d}u|=\left|\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y\right|=|y\Delta x+x\Delta y|\leqslant|y||\Delta x|+|x||\Delta y|\leqslant|y|\delta_x+|x|\delta_y,$$

$$|\Delta v|\approx|\mathrm{d}v|=\left|\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y\right|=\left|\frac{y\Delta x-x\Delta y}{y^2}\right|\leqslant\frac{|y||\Delta x|+|x||\Delta y|}{|y|^2}\leqslant\frac{|y|\delta_x+|x|\delta_y}{|y|^2},$$

于是有 $\delta_u=|y|\delta_x+|x|\delta_y$，$\delta_v=\frac{|y|\delta_x+|x|\delta_y}{|y|^2}$，从而有

$$\frac{\delta_u}{|u|}=\frac{|y|\delta_x+|x|\delta_y}{|xy|}=\frac{\delta_x}{|x|}+\frac{\delta_y}{|y|},\quad \frac{\delta_v}{|v|}=\frac{1}{\left|\frac{x}{y}\right|}\cdot\frac{|y|\delta_x+|x|\delta_y}{|y|^2}=\frac{\delta_x}{|x|}+\frac{\delta_y}{|y|}.$$

即乘积的相对误差等于各因子的相对误差之和；商的相对误差等于被除数及除数的相对误差之和.

习题 9-4

1. 设 $z=u^2+v^2$，而 $u=x+y, v=x-y$，求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

解 $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=2u\cdot 1+2v\cdot 1=4x,$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=2u\cdot 1+2v\cdot(-1)=4y\,.$$

2. 设 $z=u^2\ln v$,而$u=\dfrac{x}{y}, v=3x-2y$, 求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=2u\ln v\cdot\frac{1}{y}+\frac{u^2}{v}\cdot 3=\frac{2x}{y^2}\ln(3x-2y)+\frac{3x^2}{(3x-2y)y^2}\,,$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=2u\ln v\cdot\left(-\frac{x}{y^2}\right)+\frac{u^2}{v}\cdot(-2)=-\frac{2x^2}{y^3}\ln(3x-2y)-\frac{2x^2}{(3x-2y)y^2}\,.$$

3. 设 $z=\mathrm{e}^{x-2y}$, 而$x=\sin t, y=t^3$, 求$\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解 $$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^{x-2y}\cos t+\mathrm{e}^{x-2y}\cdot(-2)\cdot 3t^2=\mathrm{e}^{\sin t-2t^3}(\cos t-6t^2)\,.$$

4. 设 $z=\arcsin(x-y)$, 而$x=3t, y=4t^3$, 求$\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解 $$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{\sqrt{1-(x-y)^2}}\cdot 3+\frac{(-1)}{\sqrt{1-(x-y)^2}}\cdot 12t^2=\frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}\,.$$

5. 设 $z=\arctan(xy)$, 而 $y=\mathrm{e}^x$, 求$\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

解 $$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{1+x^2y^2}+\frac{x\mathrm{e}^x}{1+x^2y^2}=\frac{(1+x)\mathrm{e}^x}{1+x^2\mathrm{e}^{2x}}\,.$$

6. 设 $u=\dfrac{\mathrm{e}^{ax}(y-z)}{a^2+1}$, 而 $y=a\sin x, z=\cos x$, 求$\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

解 $$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\partial u}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{a\mathrm{e}^{ax}(y-z)}{a^2+1}+\frac{\mathrm{e}^{ax}}{a^2+1}\cdot a\cos x-\frac{\mathrm{e}^{ax}}{a^2+1}\cdot(-\sin x)$$

$$=\frac{\mathrm{e}^{ax}}{a^2+1}(a^2\sin x-a\cos x+a\cos x+\sin x)=\mathrm{e}^{ax}\sin x\,.$$

7. 设 $z=\arctan\dfrac{x}{y}$, 而 $x=u+v, y=u-v$, 验证$\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}=\dfrac{u-v}{u^2+v^2}$.

证明 $$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right)+\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\right)$$

$$=\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}+\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right)+\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}-\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right)$$

$$=\frac{2y}{x^2+y^2}=\frac{u-v}{u^2+v^2}\,.$$

8. 求下列函数的一阶偏导数（其中 f 具有一阶连续偏导数）.

(1) $u=f(x^2-y^2,\mathrm{e}^{xy})$; (2) $u=f\left(\dfrac{x}{y},\dfrac{y}{z}\right)$; (3) $u=f(x,xy,xyz)$.

解 (1) 令 $s=x^2-y^2$，$t=\mathrm{e}^{xy}$，则函数 $u=f(x^2-y^2,\mathrm{e}^{xy})$ 可视为由函数 $u=f(s,t)$ 及 $s=x^2-y^2$, $t=\mathrm{e}^{xy}$ 复合而成, 根据复合函数的求导法则, 得

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}=2xf_1'+y\mathrm{e}^{xy}f_2',$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}=-2yf_1'+x\mathrm{e}^{xy}f_2'.$$

(2) 令 $s=\dfrac{x}{y}$，$t=\dfrac{y}{z}$，则 $u=f\left(\dfrac{x}{y},\dfrac{y}{z}\right)$ 可视为由 $u=f(s,t)$ 及 $s=\dfrac{x}{y},t=\dfrac{y}{z}$ 复合而成, 根据复合函数的求导法则, 得

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}=\frac{1}{y}f_1',$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}=-\frac{x}{y^2}f_1'+\frac{1}{z}f_2',$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}=-\frac{y}{z^2}f_2'.$$

(3) 令 $s=x,t=xy,r=xyz$，则 $u=f(x,xy,xyz)$ 可视为由函数 $u=f(s,t,r)$ 及 $s=x,t=xy$, $r=xyz$ 复合而成, 根据复合函数的求导法则, 得

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}+\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}=f_1'+yf_2'+yzf_3',$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}+\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}=xf_2'+xzf_3',$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}=xyf_3'.$$

9. 设 $z=xy+xF(u)$, 而 $u=\dfrac{y}{x}$, $F(u)$ 为可导函数, 证明

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z+xy.$$

证明 $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\left[y+F(u)+xF'(u)\frac{\partial u}{\partial x}\right]+y\left[x+xF'(u)\frac{\partial u}{\partial y}\right]$$

$$=x\left[y+F(u)-\frac{y}{x}F'(u)]+y[x+F'(u)\right]=xy+xF(u)+xy=z+xy.$$

10. 设 $z=\dfrac{y}{f(x^2-y^2)}$，其中 $f(u)$ 为可导函数, 验证

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}.$$

证明 令 $u=x^2-y^2$，因为 $\dfrac{\partial z}{\partial x}=\dfrac{-yf'(u)\cdot 2x}{f^2(u)}=-\dfrac{2xyf'(u)}{f^2(u)}$，

$$\frac{\partial z}{\partial y}=\frac{f(u)+yf'(u)\cdot 2y}{f^2(u)}=\frac{f(u)+2y^2f'(u)}{f^2(u)},$$

所以 $$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{-2yf'(u)}{f^2(u)}+\frac{f(u)+2y^2f'(u)}{yf^2(u)}=\frac{1}{yf(u)}=\frac{z}{y^2}.$$

11. 设 $z=f(x^2+y^2)$，其中 f 具有二阶导数，求 $\dfrac{\partial^2 z}{\partial x^2},\dfrac{\partial^2 z}{\partial x\partial y},\dfrac{\partial^2 z}{\partial y^2}$.

解 令 $u=x^2+y^2$，则 $z=f(x^2+y^2)$ 可视为由函数 $z=f(u)$ 和 $u=x^2+y^2$ 复合而成，根据复合函数求导法则，得

$$\frac{\partial z}{\partial x}=\frac{\mathrm{d}z}{\mathrm{d}u}\cdot\frac{\partial u}{\partial x}=2xf',\quad \frac{\partial z}{\partial y}=\frac{\mathrm{d}z}{\mathrm{d}u}\cdot\frac{\partial u}{\partial y}=2yf',$$

所以 $$\frac{\partial^2 z}{\partial x^2}=2f'+2xf''\cdot\frac{\partial u}{\partial x}=2f'+4x^2f'',$$

$$\frac{\partial^2 z}{\partial x\partial y}=2xf''\cdot\frac{\partial u}{\partial y}=4xyf'',$$

$$\frac{\partial^2 z}{\partial y^2}=2f'+2yf''\cdot\frac{\partial u}{\partial y}=2f'+4y^2f''.$$

***12**. 求下列函数的 $\dfrac{\partial^2 z}{\partial x^2},\dfrac{\partial^2 z}{\partial x\partial y},\dfrac{\partial^2 z}{\partial y^2}$（其中 f 具有二阶连续偏导数）.

(1) $z=f(xy,y)$； (2) $z=f\left(x,\dfrac{x}{y}\right)$；

(3) $z=f(xy^2,x^2y)$； (4) $z=f(\sin x,\cos y,\mathrm{e}^{x+y})$.

解 (1) 令 $s=xy$，$t=y$，则 $z=f(xy,y)$ 可视为由 $z=f(s,t)$ 及 $s=xy,t=y$ 复合而成，根据复合函数的求导法则，得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial x}=yf_1',\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial y}=xf_1'+f_2',$$

这里 f_1',f_2' 仍然是以 s,t 为中间变量的关于变量 x,y 的函数. 所以

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(yf_1')=yf_{11}''\cdot\frac{\partial s}{\partial x}=y^2f_{11}'',$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(yf_1')=f_1'+y\left(f_{11}''\cdot\frac{\partial s}{\partial y}+f_{12}''\cdot\frac{\mathrm{d}t}{\mathrm{d}y}\right)=f_1'+xyf_{11}''+yf_{12}'',$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(xf_1'+f_2')=x\left(f_{11}''\cdot\frac{\partial s}{\partial y}+f_{12}''\cdot\frac{\mathrm{d}t}{\mathrm{d}y}\right)+f_{21}''\cdot\frac{\partial s}{\partial y}+f_{22}''\cdot\frac{\mathrm{d}t}{\mathrm{d}y}=x^2f_{11}''+2xf_{12}''+f_{22}''.$$

注：因为 f 具有二阶连续偏导数，所以 $f''_{12}=f''_{21}$.

(2) 令 $s=x$，$t=\dfrac{x}{y}$，则 $z=f\left(x,\dfrac{x}{y}\right)$ 可视为由 $z=f(s,t)$ 和 $s=x$，$t=\dfrac{x}{y}$ 复合而成，根据复合函数的求导法则，得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x}=f_1'+\frac{1}{y}f_2',\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial y}=-\frac{x}{y^2}f_2',$$

这里 f_1',f_2' 仍然是以 s,t 为中间变量的关于变量 x,y 的函数. 所以

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(f_1'+\frac{1}{y}f_2')=f''_{11}\cdot\frac{\partial s}{\partial x}+f''_{12}\cdot\frac{\partial t}{\partial x}+\frac{1}{y}\left(f''_{21}\cdot\frac{\partial s}{\partial x}+f''_{22}\cdot\frac{\partial t}{\partial x}\right)=f''_{11}+\frac{2}{y}f''_{12}+\frac{1}{y^2}f''_{22},$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(f_1'+\frac{1}{y}f_2'\right)=f''_{12}\cdot\frac{\partial t}{\partial y}-\frac{1}{y^2}f_2'+\frac{1}{y}f''_{22}\cdot\frac{\partial t}{\partial y}=-\frac{x}{y^2}f''_{12}-\frac{1}{y^2}f_2'-\frac{x}{y^3}f''_{22},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(-\frac{x}{y^2}f_2'\right)=\frac{2x}{y^3}f_2'-\frac{x}{y^2}f''_{22}\cdot\frac{\partial t}{\partial y}=\frac{2x}{y^3}f_2'+\frac{x^2}{y^4}f''_{22}.$$

(3) 令 $s=xy^2$，$t=x^2y$，则 $z=f(xy^2,x^2y)$ 可视为由 $z=f(s,t)$ 和 $s=xy^2$，$t=x^2y$ 复合而成，根据复合函数的求导法则，得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x}=y^2f_1'+2xyf_2',$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial y}=2xyf_1'+x^2f_2',$$

所以

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=\frac{\partial}{\partial x}(y^2f_1'+2xyf_2')=y^2\left(f''_{11}\cdot\frac{\partial s}{\partial x}+f''_{12}\cdot\frac{\partial t}{\partial x}\right)+2yf_2'+2xy\left(f''_{21}\cdot\frac{\partial s}{\partial x}+f''_{22}\cdot\frac{\partial t}{\partial x}\right)\\&=y^2(y^2f''_{11}+2xyf''_{12})+2yf_2'+2xy(y^2f''_{21}+2xyf''_{22})\\&=y^4f''_{11}+4xy^3f''_{12}+2yf_2'+4x^2y^2f''_{22}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y}&=\frac{\partial}{\partial y}(y^2f_1'+2xyf_2')=2yf_1'+y^2\left(f''_{11}\cdot\frac{\partial s}{\partial y}+f''_{12}\cdot\frac{\partial t}{\partial y}\right)+2xf_2'+2xy\left(f''_{21}\cdot\frac{\partial s}{\partial y}+f''_{22}\cdot\frac{\partial t}{\partial y}\right)\\&=2yf_1'+y^2(2xyf''_{11}+x^2f''_{12})+2yf_2'+2xy(2xyf''_{21}+x^2f''_{22})\\&=2yf_1'+2xf_2'+2xy^3f''_{11}+5x^2y^2f''_{12}+2x^3yf''_{22}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2}&=\frac{\partial}{\partial y}(2xyf_1'+x^2f_2')=2xf_1'+2xy\left(f''_{11}\cdot\frac{\partial s}{\partial y}+f''_{12}\cdot\frac{\partial t}{\partial y}\right)+x^2\left(f''_{21}\cdot\frac{\partial s}{\partial y}+f''_{22}\cdot\frac{\partial t}{\partial y}\right)\\&=2xf_1'+2xy(2xyf''_{11}+x^2f''_{12})+x^2(2xyf''_{21}+x^2f''_{22})\\&=2xf_1'+4x^2y^2f''_{11}+4x^3yf''_{12}+x^4f''_{22}.\end{aligned}$$

(4) 令 $s=\sin x,t=\cos y,r=\mathrm{e}^{x+y}$，则 $z=f(\sin x,\cos y,\mathrm{e}^{x+y})$ 可视为由函数 $z=f(s,t,r)$ 及 $s=\sin x,t=\cos y,r=\mathrm{e}^{x+y}$ 复合而成，根据复合函数的求导法则，得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial s}\cdot\frac{\mathrm{d}s}{\mathrm{d}x}+\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x}=\cos xf_1'+\mathrm{e}^{x+y}f_3',$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial t}\cdot\frac{\mathrm{d}t}{\mathrm{d}y}+\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y}=-\sin y f_2'+\mathrm{e}^{x+y}f_3',$$

所以

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(\cos x f_1'+\mathrm{e}^{x+y}f_3')$$

$$=-\sin x f_1'+\cos x\left(f_{11}''\cdot\frac{\mathrm{d}s}{\mathrm{d}x}+f_{13}''\cdot\frac{\partial r}{\partial x}\right)+\mathrm{e}^{x+y}f_3'+\mathrm{e}^{x+y}\left(f_{31}''\cdot\frac{\mathrm{d}s}{\mathrm{d}x}+f_{33}''\cdot\frac{\partial r}{\partial x}\right)$$

$$=-\sin x f_1'+\cos x(\cos x f_{11}''+\mathrm{e}^{x+y}f_{13}'')+\mathrm{e}^{x+y}f_3'+\mathrm{e}^{x+y}(\cos x f_{31}''+\mathrm{e}^{x+y}f_{33}'')$$

$$=\mathrm{e}^{x+y}f_3'-\sin x f_1'+\cos^2 x f_{11}''+2\mathrm{e}^{x+y}\cos x f_{13}''+\mathrm{e}^{2(x+y)}f_{33}''$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(\cos x f_1'+\mathrm{e}^{x+y}f_3')$$

$$=\cos x\left(f_{12}''\cdot\frac{\mathrm{d}t}{\mathrm{d}y}+f_{13}''\cdot\frac{\partial r}{\partial y}\right)+\mathrm{e}^{x+y}f_3'+\mathrm{e}^{x+y}\left(f_{32}''\cdot\frac{\mathrm{d}t}{\mathrm{d}y}+f_{33}''\cdot\frac{\partial r}{\partial y}\right)$$

$$=\cos x(-\sin y f_{12}''+\mathrm{e}^{x+y}f_{13}'')+\mathrm{e}^{x+y}f_3'+\mathrm{e}^{x+y}(-\sin y f_{32}''+\mathrm{e}^{x+y}f_{33}'')$$

$$=\mathrm{e}^{x+y}f_3'-\cos x\sin y f_{12}''+\mathrm{e}^{x+y}\cos x f_{13}''-\mathrm{e}^{x+y}\sin y f_{32}''+\mathrm{e}^{2(x+y)}f_{33}''$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(-\sin y f_2'+\mathrm{e}^{x+y}f_3')$$

$$=-\cos y f_2'-\sin y\left(f_{22}''\cdot\frac{\mathrm{d}t}{\mathrm{d}y}+f_{23}''\cdot\frac{\partial r}{\partial y}\right)+\mathrm{e}^{x+y}f_3'+\mathrm{e}^{x+y}\left(f_{32}''\cdot\frac{\mathrm{d}t}{\mathrm{d}y}+f_{33}''\cdot\frac{\partial r}{\partial y}\right)$$

$$=-\cos y f_2'-\sin y(-\sin y f_{22}''+\mathrm{e}^{x+y}f_{23}'')+\mathrm{e}^{x+y}f_3'+\mathrm{e}^{x+y}(-\sin y f_{32}''+\mathrm{e}^{x+y}f_{33}'')$$

$$=\mathrm{e}^{x+y}f_3'-\cos y f_2'+\sin^2 y f_{22}''-2\mathrm{e}^{x+y}\sin y f_{23}''+\mathrm{e}^{2(x+y)}f_{33}''.$$

***13**. 设 $u=f(x,y)$ 的所有二阶偏导数连续，而

$$x=\frac{s-\sqrt{3}t}{2},\ y=\frac{\sqrt{3}s+t}{2},$$

证明 $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2$ 及 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}$.

证明　因为 $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}=\frac{1}{2}\frac{\partial u}{\partial x}+\frac{\sqrt{3}}{2}\frac{\partial u}{\partial y}$，$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}=-\frac{\sqrt{3}}{2}\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial u}{\partial y}$，

所以 $\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2=\left(\frac{1}{2}\frac{\partial u}{\partial x}+\frac{\sqrt{3}}{2}\frac{\partial u}{\partial y}\right)^2+\left(-\frac{\sqrt{3}}{2}\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$.

又因为 $\frac{\partial^2 u}{\partial s^2}=\frac{\partial}{\partial s}\left(\frac{1}{2}\frac{\partial u}{\partial x}+\frac{\sqrt{3}}{2}\frac{\partial u}{\partial y}\right)$

$$=\frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial s}\right)+\frac{\sqrt{3}}{2}\left(\frac{\partial^2 u}{\partial y\partial x}\frac{\partial x}{\partial s}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial s}\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\frac{\sqrt{3}}{2}\frac{\partial^2 u}{\partial x\partial y}\right)+\frac{\sqrt{3}}{2}\left(\frac{1}{2}\frac{\partial^2 u}{\partial y\partial x}+\frac{\sqrt{3}}{2}\frac{\partial^2 u}{\partial y^2}\right)$$

$$=\frac{1}{4}\frac{\partial^2 u}{\partial x^2}+\frac{\sqrt{3}}{2}\frac{\partial^2 u}{\partial x\partial y}+\frac{3}{4}\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}\left(-\frac{\sqrt{3}}{2}\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial u}{\partial y}\right)$$
$$=-\frac{\sqrt{3}}{2}\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial t}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial t}\right)+\frac{1}{2}\left(\frac{\partial^2 u}{\partial y\partial x}\frac{\partial x}{\partial t}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial t}\right)$$
$$=-\frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u}{\partial x\partial y}\right)+\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\frac{\partial^2 u}{\partial y\partial x}+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}\right)$$
$$=\frac{3}{4}\frac{\partial^2 u}{\partial x^2}-\frac{\sqrt{3}}{2}\frac{\partial^2 u}{\partial x\partial y}+\frac{1}{4}\frac{\partial^2 u}{\partial y^2}.$$

所以 $\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=\frac{1}{4}\frac{\partial^2 u}{\partial x^2}+\frac{\sqrt{3}}{2}\frac{\partial^2 u}{\partial x\partial y}+\frac{3}{4}\frac{\partial^2 u}{\partial y^2}+\frac{3}{4}\frac{\partial^2 u}{\partial x^2}-\frac{\sqrt{3}}{2}\frac{\partial^2 u}{\partial x\partial y}+\frac{1}{4}\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$.

习题 9-5

1. 设 $\sin y+\mathrm{e}^x-xy^2=0$，求 $\frac{\mathrm{d}y}{\mathrm{d}x}$.

解 令 $F(x,y)=\sin y+\mathrm{e}^x-xy^2$，则

$$F_x=\mathrm{e}^x-y^2, F_y=\cos y-2xy,$$

所以 $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}=-\frac{\mathrm{e}^x-y^2}{\cos y-2xy}$.

2. 设 $\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$，求 $\frac{\mathrm{d}y}{\mathrm{d}x}$.

解 令 $F(x,y)=\ln\sqrt{x^2+y^2}-\arctan\frac{y}{x}$，则

$$F_x=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{x}{\sqrt{x^2+y^2}}-\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)=\frac{x+y}{x^2+y^2},$$

$$F_y=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{y}{\sqrt{x^2+y^2}}-\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}=\frac{y-x}{x^2+y^2},$$

所以 $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}=\frac{x+y}{x-y}$.

3. 设 $x+2y+z-2\sqrt{xyz}=0$，求 $\frac{\partial z}{\partial x}$ 及 $\frac{\partial z}{\partial y}$.

解 令 $F(x,y,z)=x+2y+z-2\sqrt{xyz}$，则

$$F_x=1-\frac{yz}{\sqrt{xyz}},\quad F_y=2-\frac{xz}{\sqrt{xyz}},\quad F_z=1-\frac{xy}{\sqrt{xyz}},$$

所以

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{yz-\sqrt{xyz}}{\sqrt{xyz}-xy}\ ,\ \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{xz-2\sqrt{xyz}}{\sqrt{xyz}-xy}.$$

4. 设 $\frac{x}{z}=\ln\frac{z}{y}$，求 $\frac{\partial z}{\partial x}$ 及 $\frac{\partial z}{\partial y}$.

解 令 $F(x,y,z)=\frac{x}{z}-\ln\frac{z}{y}$，则

$$F_x=\frac{1}{z}，\ F_y=-\frac{y}{z}\cdot\left(-\frac{z}{y^2}\right)=\frac{1}{y}，\ F_z=-\frac{x}{z^2}-\frac{y}{z}\cdot\frac{1}{y}=-\frac{x+z}{z^2},$$

所以 $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{z}{x+z}$，$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{z^2}{y(x+z)}$.

5. 设 $2\sin(x+2y-3z)=x+2y-3z$，证明 $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$.

证明 令 $F(x,y,z)=2\sin(x+2y-3z)-x-2y+3z$，则

$$F_x=2\cos(x+2y-3z)-1，\ F_y=4\cos(x+2y-3z)-2=2F_x,$$

$$F_y=2\cos(x+2y-3z)\cdot(-3)+3=-3F_x,$$

所以 $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=-\frac{F_x}{F_z}-\frac{F_y}{F_z}=\frac{1}{3}+\frac{2}{3}=1$.

6. 设 $x=x(y,z),y=y(x,z),z=z(x,y)$ 都是由方程 $F(x,y,z)=0$ 所确定的具有连续偏导数的函数，证明

$$\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x}=-1.$$

证明 因为 $\frac{\partial x}{\partial y}=-\frac{F_y}{F_x}$，$\frac{\partial y}{\partial z}=-\frac{F_z}{F_y}$，$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$，

所以 $\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x}=\left(-\frac{F_y}{F_x}\right)\cdot\left(-\frac{F_z}{F_y}\right)\cdot\left(-\frac{F_x}{F_z}\right)=-1$.

7. 设 $\phi(u,v)$ 具有连续偏导数，证明由方程 $\phi(cx-az,cy-bz)=0$ 所确定的函数 $z=f(x,y)$ 满足 $a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c$.

证明 令 $u=cx-az$，$v=cy-bz$，则函数 $\phi(cx-az,cy-bz)$ 可视为由函数 $\phi(u,v)$ 和 $u=cx-az$，$v=cy-bz$ 复合而成，所以

$$\phi_x=\phi_u\cdot\frac{\partial u}{\partial x}=c\phi_u，\ \phi_y=\phi_v\cdot\frac{\partial v}{\partial y}=c\phi_v，\ \phi_z=\phi_u\cdot\frac{\partial u}{\partial z}+\phi_v\cdot\frac{\partial v}{\partial z}=-a\phi_u-b\phi_v,$$

所以

$$\frac{\partial z}{\partial x}=-\frac{\phi_x}{\phi_z}=\frac{c\phi_u}{a\phi_u+b\phi_v},\ \frac{\partial z}{\partial x}=-\frac{\phi_x}{\phi_z}=\frac{c\phi_u}{a\phi_u+b\phi_v},$$